NATIONAL ADVISORY COMMITTEE FOR AERONAUTICS

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ANALYTIC DETERMINATION OF THE DISCHARGE COEFFICIENTS OF FLOW NOZZLES

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SUMMARY

Integration of the velocity profile at the throat of a flow nozzle yields the discharge coefficient as a function of the ratio of boundary-layer thickness to the nozzle diameter. This ratio is obtained as a solution of the approximate momentum equation for the boundary layer. The resulting expression for the discharge coefficient is then a function of the Reynolds number based on nozzle diameter and of the geometry of the nozzle. Good agreement is shown between this expression and published experimental data on flow nozzles for Reynolds numbers between 10^4 and 10^6.

INTRODUCTION

When an ideal or frictionless fluid passes through a nozzle, the flow rate is a function only of the pressure drop, fluid properties, and nozzle geometry. For the measurement of the flow rate of an actual fluid, this functional relation must be modified to include the effects of friction. This is usually done by introducing a "discharge coefficient", which is defined as the ratio of the actual mass-flow rate to the ideal mass-flow rate. Evaluation of this coefficient has been an experimental problem, and the results have usually been presented in the form of an empirical curve showing the discharge coefficient as a function of the Reynolds number.

In rounded-approach nozzles with discharge coefficients close to unity, the frictional effects are concentrated in the boundary layer. A method of obtaining an analytical relation among the discharge coefficient, Reynolds number, and the nozzle geometry by utilization of elementary boundary-layer theory is presented herein.

ANALYSIS

The actual rate of mass flow of a fluid through a nozzle is

$$W_a = \int \rho u \; dA \tag{1}$$

The integration will be taken over the cross-sectional area at the plane through the point of downstream pressure measurement, this plane hereinafter being termed the "nozzle exit". All symbols are defined in appendix A; the notation is illustrated in figure 1.

The ideal mass-flow rate that would exist if the flow were completely isentropic is

$$W_i = \rho u_{o,n} A_n \tag{2}$$

The discharge coefficient for the case where the density is constant through the cross section is defined by

$$C = \frac{W_a}{W_i} = \frac{1}{A_n} \int \frac{u}{u_{o,n}} \, dA \tag{3}$$

In order to evaluate this integral, the velocity profile $u/u_{o,n}$ must be known over the cross-sectional area. If it is assumed that the boundary-layer thickness is small compared with the radius of the nozzle, then the exact shape of the velocity profile in the boundary layer need not be known. For integration of equation (3), any of several functions could be used as approximations. The Blasius solution (ref. 2) is the best known of these. However, for mathematical convenience in appendix B, which deals with the flow of gases with heat transfer, the function

$$\frac{u}{u_o} = \tanh\,(ay)$$

was selected and will therefore be used throughout this analysis. It will be shown later to be a sufficiently close approximation to the velocity profile of a laminar boundary layer for purposes of this investigation.

A boundary-layer thickness δ is defined by

$$y = \delta \quad \text{for} \quad u/u_o = 1 - \varepsilon \quad \text{where} \quad \varepsilon << 1$$

so that

$$\frac{u}{u_o} = \tanh\left(b\frac{y}{\delta}\right) \quad \text{where} \quad b = \tanh^{-1}(1-\varepsilon) \tag{4}$$

Upon substitution of equation (4), written for the nozzle exit, equation (3) may be integrated; however, the same result is more easily obtained by introduction of the displacement boundary-layer thickness δ^*, which is defined by the relation

$$\delta^* = \int_0^\infty \left(1 - \frac{u}{u_o}\right) dy \tag{5}$$

Since it is assumed that $\delta^* << r$, the boundary-layer build-up in a nozzle may be treated as a two-dimensional flow. The boundary layer produces a blocking effect equivalent to the reduction in area caused by the displacement thickness δ^*, so that the effective exit area of a nozzle with radius r is

$$A_{e,n} = \pi(r - \delta_n^*)^2 \approx \pi r^2\left(1 - 2\,\frac{\delta_n^*}{r}\right) \tag{6}$$

The discharge coefficient is therefore given by the relation

$$C = 1 - 2\,\frac{\delta_n^*}{r} \tag{7}$$

Substitution of equation (4) in equation (5) yields upon integration

$$\delta^* = \frac{\delta}{b} \ln 2 \tag{8}$$

Therefore,

$$C = 1 - \frac{2 \ln 2}{b}\,\frac{\delta_n}{r} \tag{9}$$

The ratio δ_n/r, of course, depends on the flow conditions and nozzle geometry. Consider now von Kármán's momentum equation for the boundary layer for incompressible flow (ref. 1):

$$\rho \frac{d}{dx}\, u_o^2 \int_0^\infty \frac{u}{u_o}\left(1 - \frac{u}{u_o}\right) dy + \rho u_o \frac{du_o}{dx} \int_0^\infty \left(1 - \frac{u}{u_o}\right) dy = \tau \tag{10}$$

where u_o is the velocity just outside the boundary layer. The shearing stress at the wall τ is given by

$$\tau = \mu\left(\frac{du}{dy}\right)_{y=0}$$

Upon substituting equation (4) and performing the integrations, equation (10) becomes

$$u_o^2\delta \frac{d\delta}{dx} + \left(2 + \frac{\ln 2}{1 - \ln 2}\right)\delta^2 u_o \frac{du_o}{dx} = \frac{\mu u_o b^2}{\rho(1 - \ln 2)} \qquad (11)$$

Equation (11) may be integrated after multiplying through by $2u_o^{n-2}$ where $n = 2\left(2 + \frac{\ln 2}{1 - \ln 2}\right)$, with the result

$$\delta^2 = \frac{2\mu b^2}{\rho(1 - \ln 2)} \frac{1}{u_o^n} \int u_o^{n-1}\,dx \qquad (12)$$

The solution of equation (12) for the case where u_o = constant will be required later and is

$$\frac{\delta}{x} = \frac{b\sqrt{\frac{2}{1 - \ln 2}}}{\sqrt{Re_x}} \qquad (13)$$

Expressed in terms of the displacement thickness by substitution of equation (8), this solution is

$$\frac{\delta^*}{x} = \frac{1.77}{\sqrt{Re_x}}$$

The Blasius solution for this case (ref. 2) gives

$$\frac{\delta^*}{x} = \frac{1.73}{\sqrt{Re_x}}$$

The close agreement justifies the use of equation (4) to approximate the velocity profile. The two profiles are shown with another frequently used approximation in a dimensionless plot in figure 2.

For the general solution of equation (12) the function $u_o(x)$ is required. Consider the convergent section of the nozzle shown in figure 1. An approximation may be obtained for $u_o(x)$ for this section by assuming the flow to be one-dimensional and incompressible. For $0 < \frac{x}{l'} < \frac{\pi}{2}$ and $l' = D$, by continuity

$$u_o = u_{o,n}\left(3 - 2 \sin \frac{x}{l'}\right)^{-2} \tag{14}$$

If this expression is used, equation (12) may be numerically integrated between the limits $x/l' = 0$ and $x/l' = \pi/2$ giving

$$\frac{\delta}{l'} = \frac{0.48\ b \sqrt{\frac{2}{1 - \ln 2}}}{\sqrt{Re_{l'}}} \tag{15}$$

If the boundary layer is assumed to start at $x/l' = \pi/4$ rather than at $x/l' = 0$ and equation (12) is integrated between the limits $x/l' = \pi/4$ and $x/l' = \pi/2$, the value of the constant in equation (15) is not appreciably changed; this indicates that the flow very near the entrance has little influence on the boundary-layer growth and thereby tends to justify the use of equation (14). Moreover, if the linear relation $u_o = u_{o,n}(2x/\pi l')$ is arbitrarily assumed and substituted in equation (12), the value of the constant in equation (15) turns out to be 0.43, further indicating that accurate knowledge of the function $u_o(x)$ is not important.

The ratio δ_n/r for the entire nozzle is obtained in the following manner: The free-stream velocity u_o is assumed to vary according to equation (14) in the convergent section and is assumed to be constant and equal to $u_{o,n}$ in the straight section. The equivalent straight section of length x', which produces the same boundary-layer thickness δ_c as that produced by the convergent section, may be obtained by comparing equation (15) with equation (13) written for x', with the result

$$x' = 0.23\ l'$$

Defining $\overline{x} = x' + l$ and writing equation (13) for $\overline{x}$ yield

$$\frac{\delta_n}{l'} = \frac{b \sqrt{\frac{2}{1 - \ln 2}} \sqrt{0.23 + \frac{l}{l'}}}{\sqrt{Re_{l'}}}$$

or

$$\frac{\delta_n}{r} = \frac{2b \sqrt{\frac{2}{1 - \ln 2}}}{\sqrt{Re_D}} \sqrt{\frac{l}{D} + 0.23\ \frac{l'}{D}} \tag{16}$$

Substitution of this relation in equation (9) gives for the discharge coefficient

$$C = 1 - \frac{4(\ln 2)\sqrt{\frac{2}{1 - \ln 2}}}{\sqrt{Re_D}} \sqrt{\frac{l}{D} + \frac{1}{4}\frac{l'}{D}} \tag{17}$$

DISCUSSION

Discharge coefficients for the flow nozzles described in reference 3 were calculated from equation (17). These values are shown in figure 3 compared with the experimentally determined values reported in reference 3. In the range $10^4 < Re_D < 10^6$, the theoretical curve is well within the region of probable error of the experimental curve; this probable error in discharge coefficient ranges from ± 0.002 at $Re_D = 10^6$ to ± 0.01 at $Re_D = 10^4$. Theoretical curves obtained in reference 4 from a determination of friction factors are also shown in figure 3.

It is to be expected that equation (17) is valid for only a limited range of Reynolds numbers. For nozzles of the geometry considered in this analysis (l/D or l'/D, or both, of the order of unity) at low Reynolds numbers ($Re_D < 10^3$), equation (10) becomes less valid since it presupposes the boundary-layer thickness to be small compared with other lengths involved. At high Reynolds numbers ($Re_D > 10^6$) the boundary layer becomes turbulent, and equation (10) again becomes less valid.

Equation (10), as written, is strictly valid only for incompressible fluids; however, the terms that would be added to equation (10) to express the effects of compressibility are numerically small compared with the terms already in that equation. Moreover, these terms would affect only the boundary-layer development in the convergent section of the nozzle. Hence, compressibility should have only a small effect on the discharge coefficient, and equation (17) may be considered valid for gases as well as liquids.

Equation (17), of course, becomes inapplicable when extremes in nozzle geometries are approached: When l/D and l'/D become very large, the flow is fully developed; when l/D and l'/D are very small, the flow approaches that through a sharp-edged orifice.

For maximum accuracy in a fluid-flow measurement with a flow nozzle, a calibration of course would be required; however, equation (17) may be of value in extending a calibration curve that contains experimental discharge coefficients through only a small range of Reynolds numbers.

The analysis suggests an approach to the problem of determining changes in the discharge coefficients of flow nozzles for use in measuring the flow of gases when there is heat transfer between the gases and the nozzle surfaces. Thus, equation (3) could be written for the case where the density is variable through the cross section, by assuming a temperature profile as well as a velocity profile. A crude assumption concerning the effect of heat transfer on the boundary-layer growth would appear to be adequate to show the order of magnitude of the changes to be expected in the discharge coefficient under these conditions. The analysis for this case is presented in appendix B; however, the validity of the result has not been established by experiment.

Lewis Flight Propulsion Laboratory
National Advisory Committee for Aeronautics
Cleveland, Ohio, February 14, 1955

APPENDIX A

SYMBOLS

The following symbols are used in this report:

A	cross-sectional area of nozzle
a	$\dfrac{\tanh^{-1}(1-\varepsilon)}{\delta} = \dfrac{b}{\delta}$
b	$\tanh^{-1}(1-\varepsilon)$
C	discharge coefficient
D	diameter
l	length of straight section of nozzle
l'	axial length of convergent section of nozzle
n	$2\left(2 + \dfrac{\ln 2}{1 - \ln 2}\right)$
p	static pressure
R	gas constant
Re	Reynolds number
r	radius
t	static temperature
u	velocity
W	mass-flow rate
x	distance along nozzle surface
x'	equivalent straight length for convergent section of nozzle
$\bar{x}$	$x' + l$
y	distance perpendicular to surface of nozzle

δ boundary-layer thickness

δ^* displacement boundary-layer thickness

ε a small fraction

μ viscosity

ρ density

τ shearing stress at nozzle surface

Subscripts:

a actual

c at end of convergent section of nozzle

e effective

h with heat transfer

i ideal

n nozzle exit

o outside of boundary layer

w nozzle wall

APPENDIX B

DISCHARGE COEFFICIENTS FOR A GAS AT A TEMPERATURE DIFFERENT FROM THAT AT NOZZLE WALL

The actual rate of mass flow of a fluid through a nozzle is

$$W_a = \int \rho u \, dA \tag{B1}$$

For a gas that obeys the law

$$p = \rho R t$$

equation (B1) may be written

$$W_a = \frac{p}{R} \int \frac{u}{t} \, dA \tag{B2}$$

The ideal mass-flow rate for this case is

$$W_i = \frac{p}{R} \frac{u_{o,n}}{t_{o,n}} A_n$$

and the discharge coefficient is defined by

$$C = \frac{1}{A_n} \int \frac{u}{u_{o,n}} \frac{t_{o,n}}{t} \, dA \tag{B3}$$

For the evaluation of this integral both a velocity and a temperature profile will be assumed, with the forms

$$\frac{u}{u_o} = \tanh\,(ay) \tag{B4}$$

where $y = \delta$ for $u/u_o = 1 - \varepsilon$ and

$$\frac{t - t_w}{t_o - t_w} = \tanh\,(ay) \tag{B5}$$

where $y = 0$ for $t = t_w$ and $y = \delta$ for $\dfrac{t - t_w}{t_o - t_w} = 1 - \varepsilon$.

The assumption of the same boundary-layer thickness δ and the same shape for both the velocity and temperature profiles is essentially the Reynolds analogy. In each case the constant a, obtained by insertion of the boundary conditions, has the value

$$a = \frac{\tanh^{-1}(1-\varepsilon)}{\delta} \equiv \frac{b}{\delta}$$

Upon substitution of equations (B4) and (B5), equation (B3) may be integrated; again, however, the same result is more easily obtained by employment of a displacement boundary-layer thickness. For this case the displacement thickness δ_h^* will be defined by

$$\delta_h^* = \int_0^\infty \left(1 - \frac{u}{u_o}\frac{t_o}{t}\right) dy \tag{B6}$$

Upon substitution of equations (B4) and (B5) and integration,

$$\delta_h^* = \frac{\delta}{b} \frac{\ln 2 - \ln \frac{t_o}{t_w}}{2 - \frac{t_o}{t_w}}$$

By analogy to the case of no heat transfer,

$$C = 1 - 2 \frac{\delta_{h,n}^*}{r} \tag{B7}$$

Therefore,

$$C = 1 - \frac{2 \ln 2}{b} \frac{\delta_n}{r} \frac{1 - \frac{\ln(t_{o,n}/t_w)}{\ln 2}}{2 - t_{o,n}/t_w} \tag{B8}$$

If the assumption is made that the boundary-layer growth is not appreciably altered by the heat transfer and if the value of δ_n/r for the case of no heat transfer (eq. (16)) is inserted in equation (B8), the discharge coefficient is given by

$$C = 1 - \frac{4 \ln 2 \sqrt{\frac{2}{1 - \ln 2}}}{\sqrt{Re_D}} \sqrt{\frac{l}{D} + \frac{1}{4}\frac{l'}{D}}\, f\left(\frac{t_{o,n}}{t_w}\right) \tag{B9}$$

$$\text{where} \quad f(t_{o,n}/t_w) = \frac{1 - \dfrac{\ln(t_{o,n}/t_w)}{\ln 2}}{2 - t_{o,n}/t_w}$$

The following table shows the variation of $f(t_{o,n}/t_w)$ with $t_{o,n}/t_w$:

$t_{o,n}/t_w$	$f(t_{o,n}/t_w)$
0.1	4.92
.25	1.72
.5	1.33
1	1.00
2	.72
4	.50
10	.29

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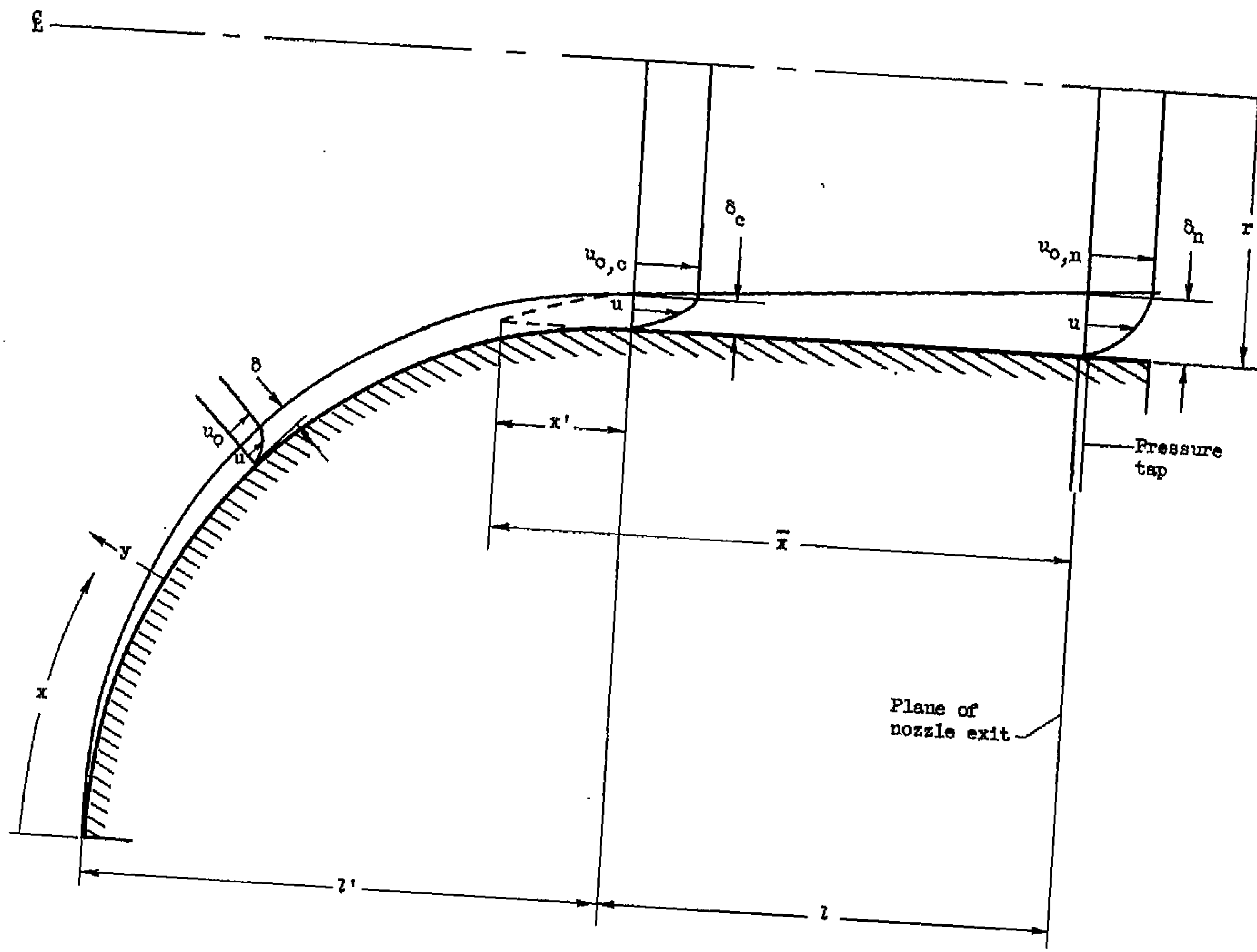

Figure 1. - Diagram of nozzle illustrating notation.

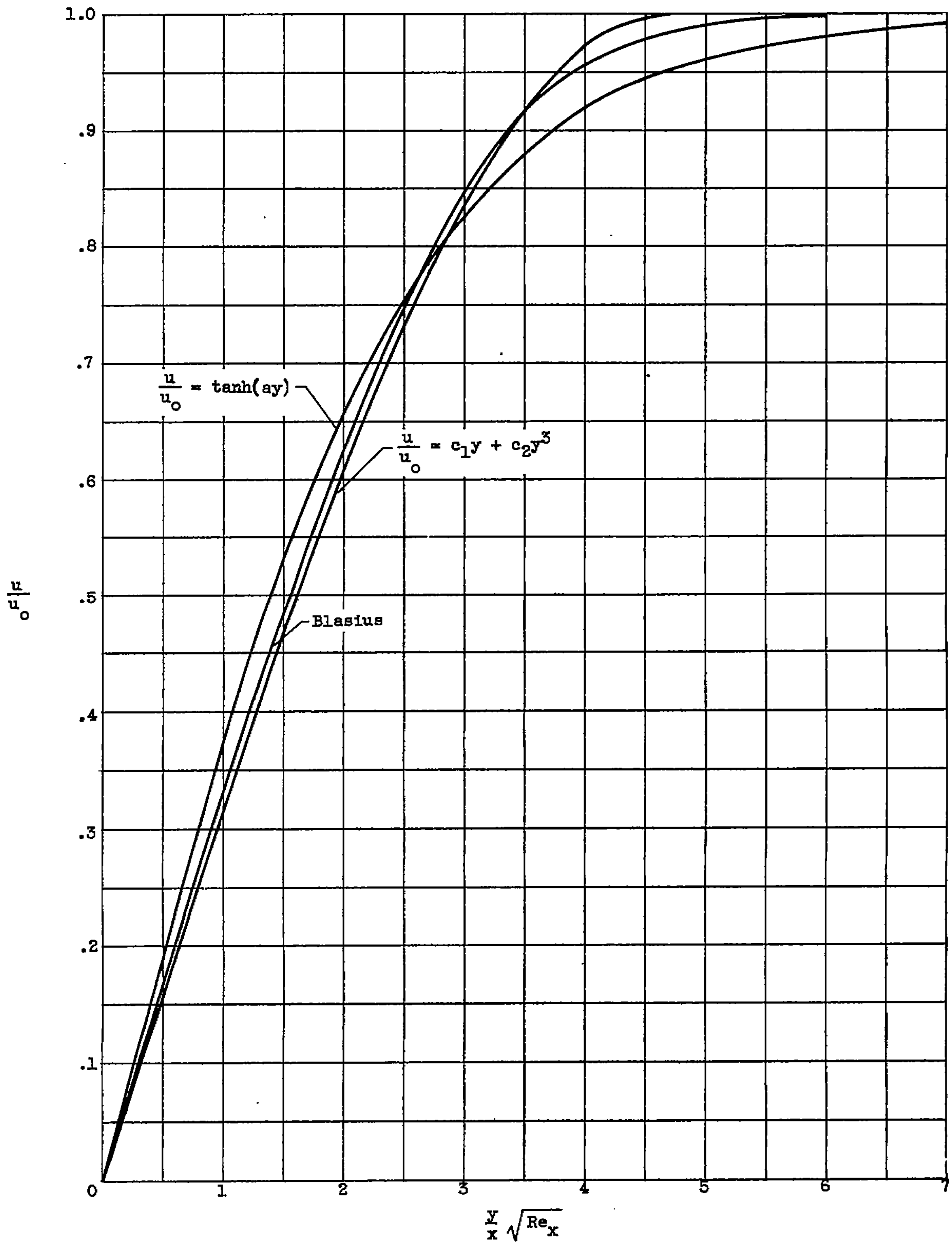

Figure 2. - Comparison of various velocity profiles.

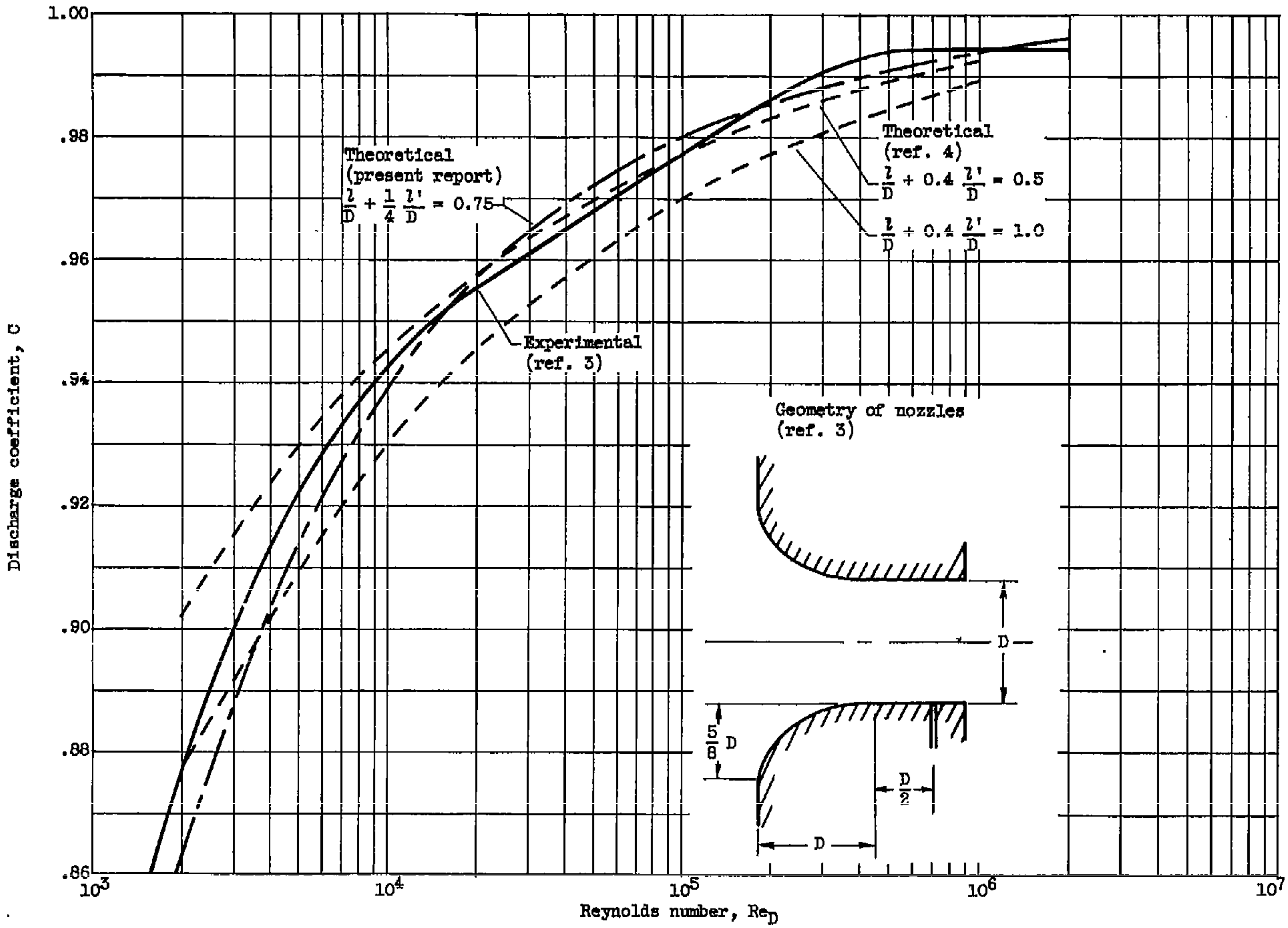

Figure 3. - Comparison of theoretical and experimental curves for discharge coefficients as functions of Reynolds number.